Preprint typeset in JHEP style - PAPER VERSION

# The Holographic Life of the $\eta'$

**José L. F. Barbón,** [a] **Carlos Hoyos,** [b] **David Mateos,** [c] **and Robert C. Myers** [c,d]

[a] *Theory Division, CERN, CH-1211, Geneva 23, Switzerland*
[b] *Instituto de Física Teórica UAM/CSIC, C-XVI*
*and Departamento de Física Teórica, C-XI*
*Universidad Autónoma de Madrid, E-28049–Madrid, Spain*
[c] *Perimeter Institute for Theoretical Physics*
*Waterloo, Ontario N2J 2W9, Canada*
[d] *Department of Physics, University of Waterloo*
*Waterloo, Ontario N2L 3G1, Canada*

*E-mail:* barbon@cern.ch, c.hoyos@uam.es, dmateos@perimeterinstitute.ca, rmyers@perimeterinstitute.ca



Abstract: In the string holographic dual of large-$N_c$ QCD with $N_f$ flavours of [1], the $\eta'$ meson is massless at infinite $N_c$ and dual to a collective fluctuation of $N_f$ D6-brane probes in a supergravity background. Here we identify the string diagrams responsible for the generation of a mass of order $N_f/N_c$, consistent with the Witten-Veneziano formula, and show that the supergravity limit of these diagrams corresponds to mixings with pseudoscalar glueballs. We argue that the dependence on the $\theta$-angle in the supergravity description occurs only through the combination $\theta+2\sqrt{N_f}\,\eta'/f_\pi$, as dictated by the $U(1)_A$ anomaly. We provide a quantitative test by computing the linear term in the $\eta'$ potential in two independent ways, with perfect agreement.

Keywords: D-branes, Supersymmetry and Duality, AdS/CFT, QCD.

## Contents



## 1. Introduction

In QCD with three light flavours of quark, $m_{\rm u}, m_{\rm d}, m_{\rm s} \ll \Lambda_{\rm QCD}$, there is a very succesful model of light meson phenomenology in terms of the spontaneous breaking of the chiral $SU(3)_{\rm L} \times SU(3)_{\rm R}$ flavour symmetry down to the diagonal subgroup. In the same context, the spontaneous breaking of the axial $U(1)_{\rm A}$ group would imply the existence of a neutral pseudoscalar meson with the quantum numbers of the $\eta'$ meson and mass $m_{\eta'} < \sqrt{3}\, m_\pi$. The measured mass of the $\eta'$ meson, close to 1 GeV, exceeds this bound by a large amount, leading to the so-called '$U(1)$ problem' [2, 3].

Quantum mechanically, the $U(1)_{\rm A}$ symmetry is broken by the anomaly, proportional to ${\rm Tr}\, F \wedge F$, which in turn means that the $U(1)$ problem is tied to the dependence of physical quantities on the $\theta$-angle of QCD. In particular, the $\eta'$ meson can only be lifted by non-perturbative effects, since the anomaly itself is a total derivative, and thus inocuous in perturbation theory.

Because of the anomaly, the effective CP-violating phase is the combination $\theta + \arg(\det m_{\rm q})$, where $m_{\rm q}$ denotes the quark mass matrix for $N_{\rm f}$ flavours. Hence, normalizing the would-be $U(1)_{\rm A}$ Goldstone boson by the global phase $e^{i\phi}$ of the $U(N_{\rm f})_{\rm A}$ Goldstone-boson matrix $\Sigma$,

the anomaly constrains the low-energy effective potential of the phase field to depend on the combination $\theta + N_{\rm f}\,\phi$ in the chiral limit, $m_{\rm q} = 0$. For example, a dilute gas of instantons generates a potential of the form (c.f. [4])

$$V(\Sigma)_{\rm inst} = A\,e^{i\theta}\,\det\,\Sigma + {\rm h.c.}\,, \tag{1.1}$$

where $A \sim \exp(-8\pi^2/g^2_{\rm YM})$. In the large-$N_{\rm c}$ limit, this potential is exponentially supressed. However, it was shown by Witten [5] (see also [6, 7]) that a non-trivial $\theta$-dependence within the $1/N_{\rm c}$ expansion of the pure Yang–Mills (YM) theory implies a potential of the form

$$V(\Sigma)_{\rm WV} = \frac{1}{2}\,\chi_{\rm YM}\,\left(\theta - i\,\log\,\det\Sigma\,\right)^2 \tag{1.2}$$

to first non-trivial order in the $1/N_{\rm c}$ expansion (generated by a non-perturbative resummation of OZI-supressed quark annihilation diagrams [8, 5, 6]). The constant $\chi_{\rm YM}$ is the topological susceptibility of the *pure* YM theory,

$$\chi_{\rm YM} = \frac{d^2\,\mathcal{E}_{\rm vac}}{d\,\theta^2}\Big|_{N_{\rm f}=0,\;\theta=0}\,, \tag{1.3}$$

to leading order in the $1/N_{\rm c}$ expansion. More generally, the large-$N_{\rm c}$ scaling of the vacuum energy density in the pure YM theory is

$$\mathcal{E}_{\rm vac} = N_{\rm c}^2\,F(\theta/N_{\rm c})\,, \tag{1.4}$$

where the function $F(y)$ has a Taylor expansion with coefficients of $O(1)$ in the large-$N_{\rm c}$ limit, and it should be multivalued under $\theta \to \theta + 2\pi$ in order for the $\theta$-angle to be defined with $2\pi$ periodicity. Then, applying the substitution $\theta \to \theta + N_{\rm f}\,\phi$ dictated by the anomaly, we find a potential of the general form

$$V(\phi) = N_{\rm c}^2\,F\left(\frac{\theta + N_{\rm f}\,\phi}{N_{\rm c}}\right)\,. \tag{1.5}$$

Notice that the multivalued nature of $\theta$-dependence in the large-$N_{\rm c}$ limit of pure YM theory is tied to an analogous 'multibranched' nature of the $\eta'$ potential, already apparent by the contrast between (1.1) and (1.2). The $\eta'$ mass is obtained by selecting the quadratic term and introducing the canonically normalized $\eta'$ field:[1]

$$\phi(x) = \frac{2}{f_\pi\,\sqrt{N_{\rm f}}}\,\eta'(x)\,, \tag{1.6}$$

where $f_\pi$ is the pion decay constant; since $f_\pi = f_{\eta'} + O(1/N_{\rm c})$, we will not distinguish between the two. This results in the famous Witten–Veneziano formula

$$m^2_{\eta'} = \frac{4N_{\rm f}}{f_\pi^2}\,\chi_{\rm YM}\,. \tag{1.7}$$

[1]Note that the present normalization is consistent with [5], however, this differs from that used in [1]: $f_\pi$[1]= $f_\pi$[5]/2.

Since $f_\pi \sim \sqrt{N_c}$, we get a mass-squared of $O(N_f/N_c)$.

In the same fashion, one can also derive soft-$\eta'$ amplitudes by applying the substitution $\theta \to \theta + 2\sqrt{N_f}\,\eta'/f_\pi$ to the $\theta$-dependence of pure-glueball amplitudes. We can specify not only the low-energy effective action of the pseudo-Goldstone field $\eta'$, but also a large-$N_c$ effective Lagrangian featuring glueballs and mesons with masses of $O(1)$ in the large-$N_c$ limit, together with a light $\eta'$ meson with mass of $O(1/N_c)$.

In string descriptions of large-$N_c$ gauge theories, such as AdS/CFT models, it should be possible to verify this scenario by direct inspection of the low-energy effective action of the string theory in the AdS-like background, either at the level of the classical supergravity approximation (glueball-meson spectrum) or at the level of string loop corrections. In particular, one should find the potential (1.5) as part of the effective action in the background geometry.

As we will review below, the first part of this check was carried out by Witten [9], who studied the $\theta$-dependence of an AdS-like model [10] dual to a non-supersymmetric, confining cousin of pure YM theory. Introducing $\theta$-dependence through Ramond–Ramond (RR) fields, Witten derived the analog of (1.4) for this model, with the result

$$\mathcal{E}^{(k)}_{\rm vac} = N_c^2\, F_k(\theta/N_c) = \frac{1}{2}\,\chi_g\,(\theta + 2\pi k)^2 + O(1/N_c) \tag{1.8}$$

to leading order in the $1/N_c$ expansion, where the integer $k$ labels the $k$-th stable 'vacuum'. Minimizing over $k$ for a given value of $\theta$ selects the true vacuum and restores the $2\pi$ periodicity. The $O(1)$ constant $\chi_g$ is the topological susceptibility in this model.

In order to complete the check we need a generalization of this setup that incorporates flavour degrees of freedom in the chiral limit. In the large-$N_c$ limit it should also incorporate a *massless*, pseudoscalar Goldstone boson that can be identified with the $\eta'$ field. Following the general ideas of [12], a model with exactly these properties was constructed in [1] by introducing flavour degrees of freedom via D6-brane probes embedded in the previous background.[2] In this note we investigate the $\eta'$ physics in this model.

We first argue that the introduction of D6-branes corresponding to massless quarks allows the dependence of the supergravity description on the microscopic $\theta$-angle to be shifted away, precisely as expected on field theory grounds. We then discuss the kind of string loop corrections that must be responsible for the generation of the anomaly-induced potential (1.2), in a string analog of the old Isgur-de Rújula-Georgi-Glashow mechanism [8]. Although we are unable to provide an independent stringy calculation of the $\eta'$ mass, we show that, in the supergravity limit, the leading Wess–Zumino coupling of the D6-brane probes to the RR background fields induces the right structure of mixings between the $\eta'$ meson and pseudoscalar glueballs. In section 4 we present a non-trivial quantitative check of this scenario by computing the linear term of the potential (1.2) in two independent ways, with precise agreement.

[2] Following the ideas of [12], meson physics has been studied in the context of AdS/CFT in [13].

In order for this paper to be self-contained, we have included, in section 2, a summary of the aspects of [1, 9, 10] that are needed in the rest of the paper. Readers who are familiar with these can go directly to section 3.

## 2. The Model

A proposal to realize a holographic dual of four-dimensional, non-supersymmetric, pure $SU(N_c)$ YM theory was made in [9]. One starts with $N_c$ D4-branes in the type IIA Minkowski vacuum $\mathbb{R}^9 \times S^1$. The D4-branes wrap the compact direction, of radius $M_{KK}^{-1}$, and anti-periodic boundary conditions are imposed for the worldvolume fermions on this circle. Before compactification, the D4-brane theory is a five-dimensional, supersymmetric $SU(N_c)$ gauge theory whose field content includes fermions and scalars in the adjoint representation of $SU(N_c)$, in addition to the gauge fields. At energies much below the compactification scale, $M_{KK}$, the theory is effectively four-dimensional. The anti-periodic boundary conditions break all of the supersymmetries and give a tree-level mass to the fermions, while the scalars also acquire a mass through one loop-effects. Thus, at sufficiently low energies, the dynamics is that of four-dimensional, massless gluons.

If the type IIA vacuum is such that there is a non-trivial holonomy around the circle for the RR one form, $C_1$, then the Wess-Zumino coupling on the D4-branes,[3]

$$\frac{1}{8\pi^2} \int_{\mathbb{R}^4 \times S^1} C_1 \wedge \mathrm{Tr}\, F \wedge F \,, \tag{2.1}$$

induces a $\theta$-term in the gauge theory with

$$\theta = \int_{S^1} C_1 \,. \tag{2.2}$$

The D4-brane system above has a dual description in terms of string theory in the near-horizon region of the associated (non-supersymmetric) supergravity background. Using this description, Witten showed [9] that the $\theta$-dependence of the vacuum energy of the YM theory has precisely the form expected on field theory grounds, as reviewed in the Introduction.

In order to explore the new physics associated to the $\eta'$ particle, we need to extend Witten's construction in such a way that, in the limit in which the KK modes would decouple, the only additional degrees of freedom would be $N_f$ flavours of fundamental, massless quarks.[4] Such an extension was proposed in [1], following the general strategy of adding fundamental matter to AdS/CFT by adding D-brane probes [12]. The construction is as follows.

[3] We adopt a nonstandard convention where the field components $(C_1)_\mu$ have dimensions of length$^{-1}$, *i.e.*, $C_1$[11]$= g_s \ell_s C_1$[present]. Hence as forms, $C_1$ and $F_2$ are both dimensionless which will simplify various expressions in the following. Note that with these conventions, the forms $C_7$ and $F_8$, defined by the usual duality relation $F_8 = * F_2$ in subsequent sections, both have dimensions of length$^6$.

[4] As usual in AdS/CFT-like dualities, this limit is not fully realisable within the supergravity approximation; see [1] for a more detailed discussion.

Consider adding $N_\mathrm{f}$ D6-branes to the original system, oriented as described by the array

$$\begin{array}{ll} N_\mathrm{c}\ \text{D4:} & 0\ 1\ 2\ 3\ 4\ \_\ \_\ \_\ \_\ \_ \\ N_\mathrm{f}\ \text{D6:} & 0\ 1\ 2\ 3\ \_\ 5\ 6\ 7\ \_\ \_\ . \end{array} \tag{2.3}$$

The original gauge fields and adjoint matter on the D4-branes arise from the light modes of the 4-4 open strings, and propagate in five dimensions. In contrast, the light modes of the 4-6 open strings give rise to $N_\mathrm{f}$ hypermultiplets in the fundamental representation of $SU(N_\mathrm{c})$ that propagate only along the four directions common to both branes.[5] Each hypermultiplet consists of one Dirac fermion, $\psi = \psi_\mathrm{L} + \psi_\mathrm{R}$, and two complex scalars. The addition of the D6-branes leaves $\mathcal{N} = 2$ unbroken supersymmetry (in four-dimensional language). This ensures that there is no force between the D4- and the D6-branes, and hence that they can be separated in the 89-plane. The bare mass of the hypermultiplets, $m_\mathrm{q}$, is proportional to this separation. If the D6-branes lie at the origin in the 89-plane, then the system enjoys a $U(1)_\mathrm{A}$ symmetry associated to rotations in this plane. A crucial fact in the construction of [1] is that, in the gauge theory, this symmetry acts on the fundamental fermions as a *chiral* symmetry, since it rotates $\psi_\mathrm{L}$ and $\psi_\mathrm{R}$ with opposite phases. Hence the $U(1)_\mathrm{A}$ symmetry acts on the relevant fields as

$$X_8 + iX_9 \to e^{i\alpha} \left( X_8 + iX_9 \right) , \qquad \psi_\mathrm{L} \to e^{i\alpha/2}\, \psi_\mathrm{L} \,, \qquad \psi_\mathrm{R} \to e^{-i\alpha/2}\, \psi_\mathrm{R} \,. \tag{2.4}$$

As discussed above, identifying the 4-direction with period $2\pi/M_\mathrm{KK}$, and with anti-periodic boundary conditions for the D4-brane fermions, breaks all of the supersymmetries and renders the theory effectively four-dimensional at energies $E \ll M_\mathrm{KK}$. Further, the adjoint fermions and scalars become massive. Similarly, we expect loop effects to induce a mass for the scalars in the fundamental representation. Generation of a mass for the fundamental fermions is, however, forbidden (in the strict large-$N_\mathrm{c}$ limit) by the existence of the chiral $U(1)_\mathrm{A}$ symmetry above. Therefore, at low energies, we expect to be left with a four-dimensional $SU(N_\mathrm{c})$ gauge theory coupled to $N_\mathrm{f}$ flavours of fundamental quark.

In the so-called 'probe limit', $N_\mathrm{f} \ll N_\mathrm{c}$, a holographic description of this theory is obtained by replacing the D4-branes by their supergravity background. The condition $N_\mathrm{f} \ll N_\mathrm{c}$ ensures that the backreaction of the D6-branes on this background is negligible, and hence that they can be treated as probes. The D6-brane worldvolume fields (and, more generally, all open string excitations on the D6-branes) are dual to gauge-invariant field theory operators constructed with at least two hypermultiplet fields, that is, meson-like operators; of particular importance here will be the quark bilinear operator, $\bar{\psi}\psi \equiv \bar{\psi}_i \psi^i$, where $i = 1, \ldots, N_\mathrm{f}$ is the flavour index.

Having reviewed the general construction, we now provide some of the details from [1] that will be needed in the following sections.

[5]We emphasize that these fields are intrinsically four-dimensional, *i.e.*, they do *not* propagate along the circle direction.

The supergravity background dual to the $N_c$ D4-branes takes the form

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu}\, dx^\mu dx^\nu + f(U) d\tau^2\right) + \left(\frac{R}{U}\right)^{3/2} \frac{dU^2}{f(U)} + R^{3/2} U^{1/2}\, d\Omega_4^2 \,, \tag{2.5}$$

$$e^\phi = g_s \left(\frac{U}{R}\right)^{3/4} , \qquad F_4 = \frac{N_c}{\Omega_4}\, \varepsilon_4 \,, \qquad f(U) = 1 - \frac{U_{\rm KK}^3}{U^3} \,. \tag{2.6}$$

The coordinates $x^\mu = \{x^0, \ldots, x^3\}$ parametrize $\mathbb{R}^4$, and correspond to the four non-compact directions along the D4-branes, as in (2.3), whereas $\tau$ parametrizes the circular 4-direction on which the branes are compactified. $d\Omega_4^2$ and $\varepsilon_4$ are the $SO(5)$-invariant line element and volume form on a unit four-sphere, respectively, and $\Omega_4 = 8\pi^2/3$ is its volume. $U$ has dimensions of length and may be thought of as a radial coordinate in the 56789-directions transverse to the D4-branes. Since the $\tau$-circle shrinks to zero size at $U = U_{\rm KK}$, to avoid a conical singularity $\tau$ must be identified with period

$$\delta\tau = \frac{4\pi}{3} \frac{R^{3/2}}{U_{\rm KK}^{1/2}} \,. \tag{2.7}$$

Under these circumstances the supergravity solution above is regular everywhere. $U$ and $\tau$ parametrize a 'cigar' (as opposed to a cylinder). That is, the surface parametrized by these coordinates is topologically a plane. The solution is specified by the string coupling constant, $g_s$, the Ramond–Ramond flux quantum (*i.e.*, the number of D4-branes), $N_c$, and the constant $U_{\rm KK}$. (The remaining parameter is given by $R^3 = \pi g_s N_c\, \ell_s^3$, with $\ell_s$ the string length.) If $U_{\rm KK}$ is set to zero, the solution (2.5, 2.6) reduces to the extremal, 1/2-supersymmetric D4-brane solution, so we may say that $U_{\rm KK}$ characterizes the deviation from extremality. The relation between these parameters and those of the $SU(N_c)$ dual gauge theory, namely, the compactification scale, $M_{\rm KK} = 2\pi/\delta\tau$, and the four-dimensional coupling constant *at* the compactification scale, $g_{\rm YM}$, is [1]:

$$R^3 = \frac{1}{2} \frac{g_{\rm YM}^2 N_c\, \ell_s^2}{M_{\rm KK}} \,, \qquad g_s = \frac{1}{2\pi} \frac{g_{\rm YM}^2}{M_{\rm KK} \ell_s} \,, \qquad U_{\rm KK} = \frac{2}{9} g_{\rm YM}^2 N_c\, M_{\rm KK} \ell_s^2 \,. \tag{2.8}$$

In the gravity description, the defining equation (2.2) for the $\theta$-angle must be understood as an asymptotic boundary condition for the RR one-form at $U \to \infty$. In other words, we must impose

$$\theta + 2\pi k = \lim_{U\to\infty} \int_{S^1} C_1 = \int_{\rm Cigar} F_2 \,, \tag{2.9}$$

where the $S^1$ is parametrized by $\tau$ and lies at $U =$ constant, as well as at constant positions in $\mathbb{R}^4$ and $S^4$, and $F_2 = dC_1$. Notice that the asymptotic holonomy of $C_1$ is measured over a contractible cycle of the background geometry. Under these circumstances, the right-hand side of (2.9) defines an arbitrary real number, and we must specify the integer $k$ to respect the angular nature of $\theta$.

To leading order in $1/N_c$, the solution of the supergravity equations that obeys the constraint (2.9) is obtained [9] simply by adding to (2.5) and (2.6) the RR two-form

$$F_2 = \frac{C}{U^4}\,(\theta + 2\pi k)\, dU \wedge d\tau\,, \tag{2.10}$$

where $C = 3\,U_{\rm KK}^3/\delta\tau$. Inserting this expression into the kinetic action of the RR forms we get Witten's result for the energy density

$$\mathcal{E}_{\rm vac}^{(k)} = \frac{1}{2(2\pi)^7 \ell_s^6 V_4} \int F_2 \wedge *F_2 = \frac{1}{2}\,\chi_g\,(\theta + 2\pi k)^2\,, \tag{2.11}$$

where $V_4 = \int d^4x$. The topological susceptibility is thus given by (c.f. [14])

$$\chi_g = \frac{(g_{\rm YM}^2 N_c)^3}{4\cdot(3\pi)^6}\, M_{\rm KK}^4\,. \tag{2.12}$$

The generation of a topological susceptibility of $O(1)$ constrasts with naive expectations based on an instanton gas picture. In this model, one can explicitly check that the semiclassical approximation based on a dilute instanton gas does not commute with the large-$N_c$ resummation provided by the supergravity approximation [15].

The study of the embedding of the D6-brane probes is greatly simplified by working in isotropic coordinates in the 56789-directions. Towards this end, we first define a new radial coordinate, $\rho$, related to $U$ by

$$U(\rho) = \left(\rho^{3/2} + \frac{U_{\rm KK}^3}{4\rho^{3/2}}\right)^{2/3}\,, \tag{2.13}$$

and then five coordinates $\vec{z} = (z^5, \ldots, z^9)$ such that $\rho = |\vec{z}|$ and $d\vec{z}\cdot d\vec{z} = d\rho^2 + \rho^2\, d\Omega_4^2$. In terms of these coordinates the metric (2.5) becomes

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu}\, dx^\mu dx^\nu + f(U) d\tau^2\right) + K(\rho)\, d\vec{z}\cdot d\vec{z}\,, \tag{2.14}$$

where

$$K(\rho) \equiv \frac{R^{3/2} U^{1/2}}{\rho^2}\,. \tag{2.15}$$

Here $U$ is now thought of as a function of $\rho$. To exploit the symmetries of the D6-brane embedding, we finally introduce spherical coordinates $\lambda, \Omega_2$ for the $z^{5,6,7}$-space and polar coordinates $r, \phi$ for the $z^{8,9}$-space. The final form of the D4-brane metric is then

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(\eta_{\mu\nu}\, dx^\mu dx^\nu + f(U) d\tau^2\right) + K(\rho)\, \left(d\lambda^2 + \lambda^2\, d\Omega_2^2 + dr^2 + r^2\, d\phi^2\right)\,, \tag{2.16}$$

where $\rho^2 = \lambda^2 + r^2$. The $U(1)_{\rm A}$ symmetry corresponds here to shifts of the $\phi$ coordinate.

In these coordinates the D6-brane embedding takes a particularly simple form. We use $x^\mu$, $\lambda$ and $\Omega_2$ (or $\sigma^a$, $a = 0, \ldots, 6$, collectively) as worldvolume coordinates. The D6-brane's

position in the 89-plane is specified as $r = r(\lambda)$, $\phi = \phi_0$, where $\phi_0$ is a constant. Note that $\lambda$ is the only variable on which $r$ is allowed to depend, by translational and rotational symmetry in the 0123- and 567-directions, respectively. We also set $\tau =$ constant, as corresponds to D6-branes localized in the circle direction.

The function $r(\lambda)$ is determined by the requirement that the equations of motion of the D6-brane in the D4-brane background be satisfied. In the supersymmetric limit, $U_{\rm KK} = 0$, $r(\lambda) = 2\pi\ell_s^2\, m_{\rm q}$ is a solution for any (constant) quark mass $m_{\rm q}$, as depicted in figure 1(a); this reflects the BPS nature of the system. If the quarks are massive then the D6-brane embedding is not invariant under rotations in the 89-plane and the $U(1)_{\rm A}$ symmetry is explicitly broken. If instead $m_{\rm q} = 0$ then the $U(1)_{\rm A}$ symmetry is preserved.

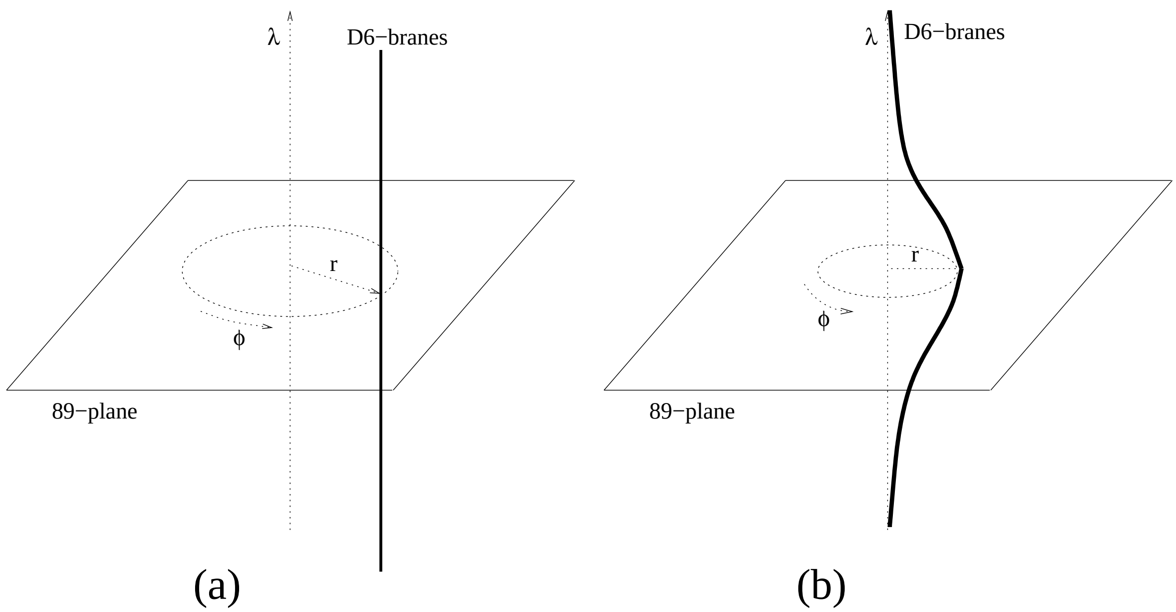


**Figure 1:** (a) D6-brane embedding if $U_{\rm KK} = 0$, for some non-zero value of $m_{\rm q}$. (b) D6-brane embedding for $U_{\rm KK} \neq 0$ and $m_{\rm q} = 0$.

If $U_{\rm KK} \neq 0$ supersymmetry is broken and $r(\lambda) =$ constant is no longer a solution. The new solution is found as follows. For large $\lambda$, the equation of motion linearizes, and its general solution is

$$r(\lambda) \simeq 2\pi\ell_s^2\, m_{\rm q} + \frac{c}{\lambda} + O(\lambda^{-2})\,. \tag{2.17}$$

As explained in [1], the field $r(\lambda)$ is dual to the quark bilinear operator $\bar{\psi}\psi$, so the constants $m_{\rm q}$ and $c$ are dual to the quark mass and the chiral condensate, respectively. The requirement that the solution be regular everywhere imposes a constraint between these two constants, that is, determines $c = c(m_{\rm q})$. This is exactly as expected on field theory grounds, since the chiral condensate should be dynamically determined once the quark mass is specified.

The solution for massless quarks is depicted in figure 1(b). We see that, although the D6-branes align asymptotically with the $\lambda$-axis, they develop a 'bump' in the 89-plane as $\lambda \to 0$, that is, $r(0) \neq 0$. The D6-brane embedding is therefore not invariant under rotations in the 89-plane, and hence the $U(1)_{\rm A}$ symmetry is *spontaneously* broken. The reason why this breakng is spontaneous is that both the boundary condition, $r(\infty) = 0$, and the D6-brane equation of motion, are $U(1)_{\rm A}$-invariant, yet the lowest-energy solution breaks the $U(1)_{\rm A}$ symmetry. On gauge theory grounds, we expect this breaking to be caused by a non-zero chiral condensate, $\langle \bar{\psi}\psi \rangle \neq 0$. This is confirmed in the gravity description by the fact that $c(m_{\rm q})$ approaches a non-zero constant in the limit $m_{\rm q} \to 0$ [1].

The D6-brane embedding described above must be thought as the 'vacuum state' of the D6-branes in the D4-brane background. By studying fluctuations around this embedding, the spectrum of (a certain class of) scalar and pseudoscalar mesons was computed in [1]. In particular, for $N_{\rm f} = 1$, a massless, pseudoscalar meson was found. This is the Goldstone boson expected from the spontaneous breaking of $U(1)_{\rm A}$ symmetry, that is, the $\eta'$. The corresponding mode in the gravity description is the zero mode associated to rotations of the D6-brane in the 89-plane, that is, it corresponds to fluctuations of the D6-brane worldvolume field $\phi$.[6]

## 3. The Anomaly-induced Potential and Glueball Mixings

In this section we discuss the general structure of $1/N_{\rm c}$ corrections responsible for the generation of a potential that lifts the $\eta'$ meson. We first show that the introduction of D6-branes corresponding to massless quarks allows the $\theta$-dependence of the supergravity description to be shifted away, as expected on field theory grounds. We then isolate the relevant string diagrams and study their main properties in the supergravity approximation.

### 3.1 The anomaly relation in the ultraviolet regime

At very high energies, the string model based on $N_{\rm c}$ D4-branes and $N_{\rm f}$ D6-branes realizes the anomalous $U(1)_{\rm A}$ symmetry of QCD as an R-symmetry on their common $\mathbb{R}^4$ worldvolume. Since this symmetry is anomalous, the $U(1)_{\rm A}$ rotation of the D6-brane fields by an angle $\alpha$, as specified in (2.4), must be equivalent to a shift of the effective $\theta$-angle in (2.1) by

$$\int_{S^1} C_1 \to \int_{S^1} C_1 + N_{\rm f}\,\alpha\,, \tag{3.1}$$

so that the dependence on the microscopic $\theta$-angle can be eliminated by a phase rotation of the $X_8 + iX_9$ field, as argued in the Introduction.

In the dual gravity description, the microscopic coupling (2.1) and the elementary quark fields $\psi_{\rm L,R}$ are not directly visible, since the D4-branes are replaced by the background (2.5) and the effective action only contains colour-singlet degrees of freedom. However, the fact

[6] The odd-parity nature of these fluctuations is due to the fact that a gauge-theory parity transformation acts on $X_8 + iX_9 = re^{i\phi}$ by complex conjugation. See [1] for a detailed discussion.

that the dependence on the microscopic $\theta$-angle can be eliminated, as implied by the anomaly, still follows from topological properties of the RR fluxes induced by the D6-branes, as we now show.

In the gravity description, the microscopic $\theta$-angle is defined by the boundary condition (2.9). The key observation is that the D6-branes' contribution to this integral has precisely the form (3.1). To see this, we recall that, by definition, the D6-branes are magnetic sources for the RR two-form, such that the flux through any two-sphere that links the D6-branes is

$$\int_{S^2} F_2 = 2\pi N_{\mathrm{f}} \, . \tag{3.2}$$

The D6-branes are localized in the $\tau$-direction, and, in the chiral limit, they are also asymptotically localized at the origin of the 89-plane, *i.e.*, $\lim_{\lambda\to\infty} r(\lambda) = 0$. A two-sphere surrounding the D6-branes in this region is shown in figure 2. Since $\tau$ is periodically identified, this two-sphere can be continously deformed to a torus, $T^2$, parametrized by $\tau$ and $\phi$ at fixed $r$ and (large) $\lambda$. Since $F_2$ is a closed form, the captured flux is the same, *i.e.*,

$$\int_{T^2} F_2 = 2\pi N_{\mathrm{f}} \, . \tag{3.3}$$

Since a translation in $\phi$ is an isometry of the background, it follows that the flux through any strip defined by two angles $\phi_1$ and $\phi_2$, as in the figure, must be proportional to the area of the strip, that is,

$$\int_{\mathrm{Strip}} F_2 = N_{\mathrm{f}}(\phi_2 - \phi_1) \, . \tag{3.4}$$

Note that this result relies crucially on the fact that all integrals above are evaluated in the UV, *i.e.*, in the limit $\lambda \to \infty$, as appropriate to the definition of the *microscopic* $\theta$-angle. In this limit the D6-branes lie at the origin of the 89-plane and the integrals above are insensitive to the deformation of the D6-branes in the region $\lambda \to 0$.

Finally, since locally we have $F_2 = dC_1$, we can use Stokes' theorem to write

$$\int_{\mathrm{Strip}} F_2 = \int_{S^1_{\phi_2}} C_1 - \int_{S^1_{\phi_1}} C_1 \, , \tag{3.5}$$

where $S^1_{\phi_i}$ is parametrized by $\tau$ at $\phi = \phi_i$. Combining these results we deduce that the Wilson line of $C_1$ at a given angle $\alpha$, as induced by the D6-branes, is

$$\int_{S^1_\alpha} C_1 = N_{\mathrm{f}}\, \alpha \, , \tag{3.6}$$

where we have set to zero a possible additive constant by choosing the origin of the polar angle $\alpha$ appropriately. If, in addition, there is a background value for this Wilson line (an asymptotically flat connection defining the $\theta$-angle) then the total value of the Wilson line is

$$\int_{S^1_\alpha} C_1 = \theta + N_{\mathrm{f}}\, \alpha \, . \tag{3.7}$$

Under a rotation by angle $\alpha$ in the 89-plane of the background, the 'Dirac sheet' singularity that is used to define $C_1$ (extending as a string in the plane $(r, \phi)$ at $\phi = 0$) rotates by minus this same angle and shifts the theta angle according to (3.7). Since the position of this Dirac sheet is a gauge artefact, we see explicitly how the microscopic $\theta$-angle can be shifted away by a $U(1)_{\rm A}$ transformation.

This supergravity argument proves that the physics is independent of the microscopic $\theta$-angle when the D6-branes are asymptotically located at the origin of the 89-plane, *i.e.*, in the chiral limit. Supersymmetry breaking at a scale $M_{\rm KK}$ implies that a shift $\delta\theta$ of the $\theta$-angle by a change of the RR two-form $F_2$ costs energy $\chi_g\, \theta\, \delta\theta$, to linear order in $\delta\theta$. At the same time, chiral symmetry breaking implies that a linear potential $\chi_g\, \theta\, N_{\rm f}\, \phi$ for the D6-brane coordinate $\phi$ must be somehow generated, so that the complete potential energy is only a function of the $U(1)_{\rm A}$-invariant combination $\theta + N_{\rm f}\phi$. This is checked in section 4 by an explicit computation.

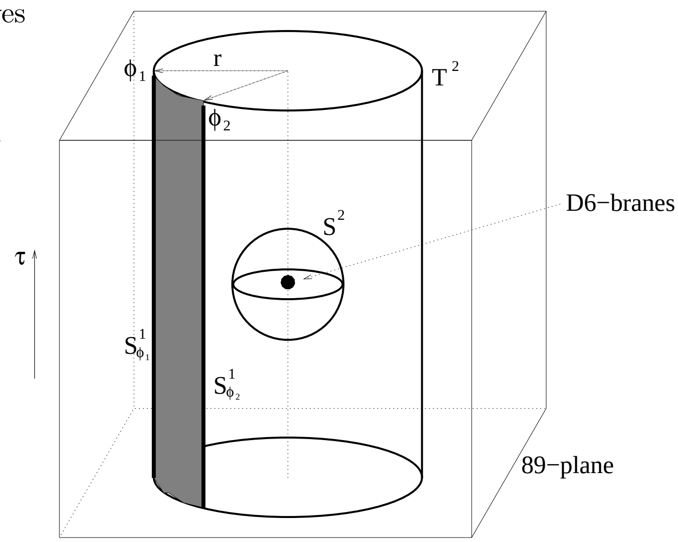


**Figure 2:** Asymptotically, the D6-branes lie at the origin of the 89-plane and are localized in the $\tau$-direction.

Since $\phi$ starts life in ten dimensions as a gauge field, the mass term $\frac{1}{2}\,\chi_g\, N_{\rm f}^2 \phi^2$ looks very much like a Green–Schwarz correction to the field-strength of the $C_1$ axion field. It would be interesting to confirm this by finding a more geometrical construction in ten-dimensional notation.

### 3.2 String contributions to the potential

The $\theta$-dependence computed by Witten in the pure-glue sector, plus the above anomaly argument, constrain the leading potential of the $\eta'$ field in the $k$-th branch to be

$$V(\phi)^{(k)} = \frac{1}{2}\, \chi_g\, (\theta + 2\pi k + N_{\rm f}\, \phi\,)^2\,. \tag{3.8}$$

Mimicking the field theory arguments of [5, 6] we can identify the candidate string diagrams that generate the mass term by considering string contributions to the two-point function of the *total* topological susceptibility $\chi_{\rm total}$, which vanishes because of the anomalous $U(1)_{\rm A}$ symmetry. In the string loop expansion, the pure-glue contribution calculated in (2.12) must be cancelled by contributions from meson diagrams.

The leading such diagram is depicted in figure 3 and features a single open-string boundary attached to the D6-branes, together with two closed-string vertex operators dual to the anomaly operator $\mathcal{Q} = \mathrm{Tr}\, F \wedge F$. This diagram is the string counterpart of the OZI-suppressed quark annhilation diagrams considered in [8, 5, 6].

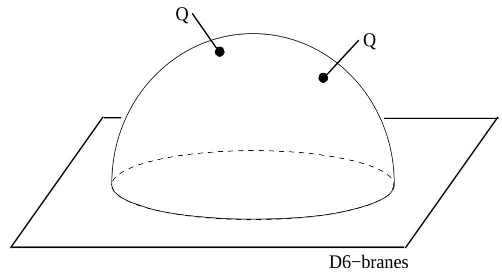


**Figure 3:** The leading open-string correction to the two-point function $U_1(p)$ of topological charge operators $\mathcal{Q} = \mathrm{Tr}\, F \wedge F$.

A spectral decomposition of this diagram yields

$$U_1(p) = \frac{N_\mathrm{f}}{N_\mathrm{c}} \sum_n \frac{|C_n|^2}{p^2 + m_n^2} \,, \tag{3.9}$$

where $C_n = O(1)$ in the large-$N_\mathrm{c}$ limit and the meson spectrum $m_n$, calculated from the fluctuations of the D6-brane, is also of $O(1)$, except for the lowest excitation, the $\eta'$, which is massless. The contribution to the topological susceptibility arises from the formal $p \to 0$ limit, which of course is infrared-divergent because of the massless $\eta'$ meson.

A standard procedure to resolve this infrared divergence is to resum a chain of highly divergent diagrams, $U_h(p)$, of the form depicted in figure 4, where the index $h$ stands for the number of open-string boundaries. Isolating the massless meson in $h$ intermediate propagators, we see that $U_h(k)$ diverges in the infrared as $(N_\mathrm{f}/N_\mathrm{c} p^2)^h$. Summing up the geometric series of such terms induces a 1PI self-energy contribution, of order $N_\mathrm{f}/N_\mathrm{c}$, given by the cylinder diagram in figure 5(a). The same diagram with the other possible inequivalent insertion of the $\eta'$ field (contributing of course self-energy corrections of the same order) is depicted in figure 5(b).

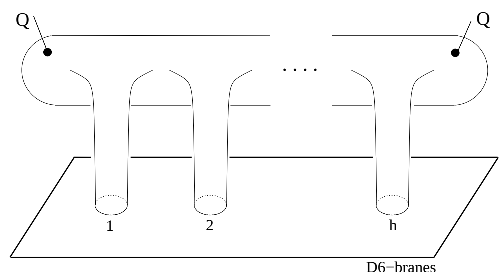


**Figure 4:** Diagram $U_h(p)$ with $h$ open-string boundaries.

The closed and open string interpretations[7] of these diagrams is given in figures 6 and 7, respectively. Note that the indices carried by the double lines are not $SU(N_\mathrm{c})$ colour indices (there are only $SU(N_\mathrm{c})$ singlets in the gravity description) but flavour indices of $SU(N_\mathrm{f})$,[8] under which the pion fields transform in the adjoint representation but the $\eta'$ is inert.

As shown by its open string representation, the diagram in figure 5(b) is equivalent to a standard one-loop correction in the effective meson theory, and this contributions is common to singlet and non-singlet mesons. In contrast, the diagram in figure 5(a) will only couple to

[7]By this we mean those obtained by cutting the diagrams in such a way that the intermediate states are closed or open strings, respectively. Of course, in both cases the external states are an open string state, namely, the $\eta'$.

[8]This is a global symmetry of the boundary field theory, and a gauge symmetry on the worldvolume of the D6-branes in the dual gravity description.

the flavour singlet mesons and so distinguishes the behaviour of the $\eta'$ meson from the rest of the 'Goldstone' modes. This would suggest that, at a quantitative level, this diagram gives the most important contribution to the mass of the $\eta'$.

In order to contribute to the $\eta'$ mass, either of these self-energy corrections must shift the zero-momentum pole of the large-$N_{\rm c}$ meson propagator. Unfortunately, direct computation of the full string diagrams is not possible in the background in question, since we are restricted to the supergravity approximation. It is then interesting to separate the part of figure 5(a) corresponding to the exchange of supergravity modes from a stringy 'contact term' coming from the infinite tower of closed string modes and possible contributions at the boundary of worldsheet moduli space. The contribution of a *finite* number of low-lying glueball modes with mass $M_n$ shifts the $\eta'$ pole mass-squared by

$$\delta m_{\eta'}^2 = -\sum_n \frac{g_n(0)^2}{M_n^2} \;, \tag{3.10}$$

where $g_n(0)$ stands for the zero-momentum limit of the glueball-$\eta'$ coupling, which must be non-vanishing for this contribution to be non-trivial. The shift (3.10) has the 'wrong' sign though, so the stringy contact term (the high-energy part of the full string diagram) must be positive and all-important at the quantitative level.

We will elaborate further on these issues in the last section.

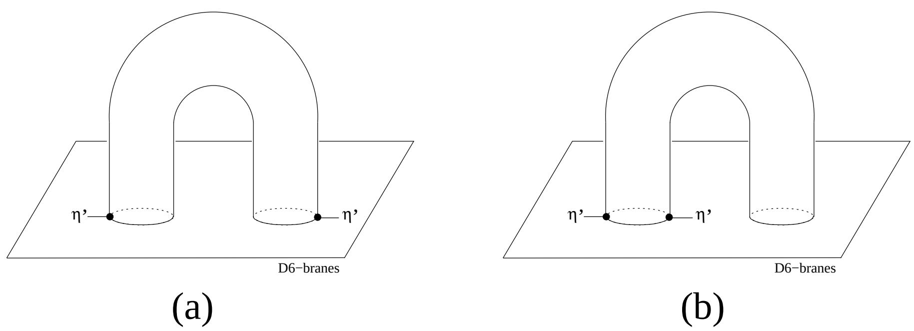


**Figure 5:** Basic cylinder diagram of order $N_{\rm f}/N_{\rm c}$, with the two possible inequivalent insertions of the $\eta'$ field.

### 3.3 Meson-glueball mixing

In this subsection we show that $g_n(0) \neq 0$ by analysing the glueball-$\eta'$ mixing at the supergravity level. Quite generally, any closed-string field $\Phi_c$ that is sourced by the D6-branes and has non-trivial wave-function with respect to the $\phi$ angle is subject to mixing with the $\eta'$ meson. Expanding $\Phi_c$ in Fourier modes one has

$$\Phi_c(\phi) = \sum_n \mathcal{G}_n \, e^{-in\phi} \;, \tag{3.11}$$

where the normalizable modes $\mathcal{G}_n$, when pulled back to the $\mathbb{R}^4$ factor in the D6 world-volume, represent glueballs of $U(1)_{\rm A}$ charge $n$. If $\Phi_c$ enters linearly the world-volume theory on the

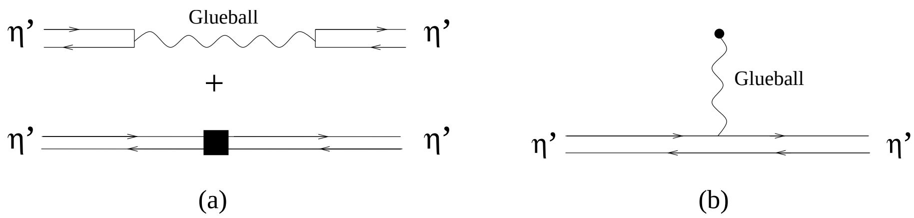


**Figure 6:** Closed string interpretation of the cylinder diagrams of figure 5. The representation (a) exhibits the fact that diagram 5(a) contributes low-lying glueball mixing at tree level (supergravity fields) plus a high-energy contact term coming from the infinite tower of closed string states. The $\eta'$-glueball coupling in (a) is of order $\sqrt{N_{\rm f}/N_{\rm c}}$. The strength of the glueball tadpole in (b) is of order $N_{\rm f}$, whereas the cubic $\eta'\eta'$-glueball vertex is of order $1/N_{\rm c}$ — see Appendix A.

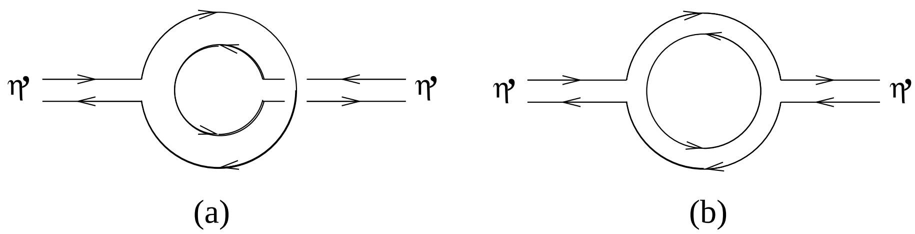


**Figure 7:** Open string interpretation of the cylinder diagrams of figure 5. (b) is the standard one-loop meson self-energy; each of the two vertices is of order $1/\sqrt{N_{\rm c}}$, and the internal loop yields a factor of $N_{\rm f}$. The internal meson propagators in (a) are both twisted. The factor of $N_{\rm f}$ now comes from the fact that the flavour of the incoming lines need not be the same as that of the outgoing lines.

D6-branes, equation (3.11) gives the required non-derivative couplings to the $\eta'$ meson. The prototypical example is the dilaton term in the Born-Infeld action:

$$\frac{1}{(2\pi)^6\,\ell_s^{\,7}}\sum_{j=1}^{N_{\rm f}}\int_{\Sigma_7} e^{-\Phi}\,\sqrt{-\det G_{\rm ind}(\partial\phi_j)}\;, \tag{3.12}$$

where $\Sigma_7$ is the D6-branes worldvolume,

$$\sqrt{-\det G_{\rm ind}\,(\partial\phi_j)} = 1 + O\left[(\partial\phi_j)^2\right] \tag{3.13}$$

and the corresponding pullbacks are understood to each of the coinciding $N_{\rm f}$ branes. Selecting the non-derivative term in the expansion of the square root we have couplings of the form (3.11). In chiral-lagrangian notation, assembling the collective coordinates of the D6-branes in a Goldstone-boson diagonal matrix $\Sigma = {\rm diag}\,(e^{i\phi_j})$, we have terms of the form

$$N_{\rm c}\sum_n \int_{\mathbb{R}^4} \mathcal{G}_n\,{\rm Tr}\,\Sigma^n + {\rm h.c.} \tag{3.14}$$

These glueballs, being charged with respect to the $U(1)_{\rm A}$ group, are ‘Kaluza–Klein artefacts’, not present in real QCD. In fact, the couplings (3.14) respect the $U(1)_{\rm A}$ symmetry and cannot

induce a potential that breaks it upon integrating out the glueballs. For example, at tree level we generate terms proportional to $\mathrm{Tr}\,\Sigma^n \cdot \mathrm{Tr}\,\Sigma^{-n}$, because the glueball propagator couples $\mathcal{G}_n$ and $\mathcal{G}_{-n}$ ($n$ being a Kaluza–Klein momentum). The global phase $e^{i\phi}$ drops from these expressions and we see that such couplings do not generate a potential for the $\eta'$ particle. This is just as well, since such contributions seem completely independent of the $\theta$-dependence, as dictated by the Witten–Veneziano formula.

In fact, the candidate glueballs are selected by the general arguments in the Introduction. First, we expect the required couplings to show the characteristic multivaluedness of $\theta$-dependence at large $N_{\rm c}$, *i.e.*, we expect the coupling to be a function of $-i\,\log\det\Sigma \sim N_{\rm f}\,\phi$, precisely linear in the angular coordinate, so that the angular periodicity of the effective action would require an explicit sum over different branches. The supposed relation to $\theta$-dependence suggests that we investigate the glueballs in the RR sector of the closed-string theory.

Natural candidates are the normalizable modes of the RR potential $C_1$, or, equivalently, of its Hodge dual $C_7$, since these give rise to *pseudo*-scalar glueballs. In the absence of D6-brane sources, it is truly equivalent to work with $C_1$ or $C_7$. The D6-branes, however, couple minimally to $C_7$ through the Wess–Zumino term

$$S_{\rm WZ} = \frac{N_{\rm f}}{(2\pi\ell_s)^6}\int_{\Sigma_7} C_7\,. \tag{3.15}$$

In terms of $C_7$, this coupling is both local and can be defined for off-shell values of the RR seven-form potential. For on-shell configurations this coupling can be reexpressed in terms of $C_1$ at the expense of introducing non-locality. However, since we wish to exhibit $\eta'$-glueball couplings at zero momentum, which are necessarily off-shell, we must work with $C_7$.

Now we wish to demonstrate that (3.15) contains a linear coupling to the $\phi$ field when reduced to the $\mathbb{R}^4$ factor of the space-time. Towards this end, let us consider the following ansatz for fluctuations: $C_7 = (\phi+\phi_0)W_7$, where $\phi_0$ is a constant and $W_7$ is the $\phi$-independent seven-form

$$\begin{aligned} W_7 = \;& -G(x)\,h(U)\,r\lambda^2\,(r d\lambda - \lambda dr)\wedge d\Omega_2\wedge dV_4 \\ & +\tilde{h}(U)\,r\lambda^2\,d\lambda\wedge dr\wedge d\Omega_2\wedge i_{N(x)}dV_4\,. \end{aligned} \tag{3.16}$$

Here $d\Omega_2$ and $dV_4$ are the volume forms on the $S^2$ wrapped by the D6-branes and on the $\mathbb{R}^4$ factor, respectively. $G(x)$ is a pseudoscalar field and $N^\mu(x)$ is a vector field with

$$i_{N(x)}dV_4 = \frac{1}{3!}N^\mu(x)\,\epsilon_{\mu\nu\alpha\beta}\,dx^\nu\wedge dx^\alpha\wedge dx^\beta\,. \tag{3.17}$$

Finally, $h(U)$ and $\tilde{h}(U)$ are radial profiles to be determined. Note that $C_7$, not being gauge-invariant, is allowed to be multivalued in $\phi$,[9] but its gauge-invariant field strength, $F_8 = dC_7$, must be single-valued. This restriction forces $W_7$ to be closed, which in turn implies

$$G(x)\,\left[5h(U) + \rho\,\frac{dU}{d\rho}\,h'(U)\right] + \partial_\mu N^\mu(x)\,\tilde{h}(U) = 0\,, \tag{3.18}$$

[9]An average over the action $\phi_0 \to \phi_0 + 2\pi$ can restore the angular character of $\phi$ that is lost in the expression for $C_7$, and is the counterpart of the average over large-$N_{\rm c}$ branches of $\theta$-dependence.

and therefore

$$\tilde{h} = 5h + \rho \frac{dU}{d\rho} h' \,, \qquad \partial_\mu N^\mu = -G \,. \tag{3.19}$$

We see that closure of $W_7$ relates the two radial functions, as well as the scalar and the vector. Under these conditions $F_8 = d\phi \wedge W_7$, which is, of course, single-valued.

We may regard the second equation above as a constraint on $N$, and $G$ as a yet totally unconstrained pseudoscalar glueball field. In fact, $N$ is not an independent field on-shell, but is completely determined by $G$. Indeed, the equations of motion for $G$ and $N$ come from $d * F_8 = 0$, or equivalently the Bianchi identity $dF_2 = 0$. A straightforward calculation yields

$$F_2 = *F_8 = G(x)\, \tilde{H}(U)\, \rho \left(\frac{dU}{d\rho}\right)^{-1} d\tau \wedge dU - H(U)\, N_\mu\, d\tau \wedge dx^\mu \,, \tag{3.20}$$

where

$$\begin{aligned} \tilde{H}(U) &= -\left(\frac{U}{R}\right)^{-9/4} K(\rho)^{-3/2}\, f(U)^{1/2}\, h(U) \,, \\ H(U) &= \left(\frac{U}{R}\right)^{-3/4} K(\rho)^{-5/2}\, f(U)^{1/2}\, \tilde{h}(U) \,, \end{aligned} \tag{3.21}$$

and we have made use of the fact that

$$dU = \frac{dU}{d\rho} \left(\frac{\lambda}{\rho}\, d\lambda + \frac{r}{\rho}\, dr\right) . \tag{3.22}$$

Closure of $F_2$ then implies

$$H' = M^2\, \tilde{H}\, \rho \left(\frac{dU}{d\rho}\right)^{-1} , \qquad N_\mu = -\frac{1}{M^2}\, \partial_\mu G \tag{3.23}$$

for some constant $M^2$. As anticipated, the second equation above determines $N$ in terms of $G$. Combined with the constraints (3.19), it imposes the on-shell condition for $G$:

$$\partial_\mu \partial^\mu\, G = M^2\, G \,. \tag{3.24}$$

Further combining the first constraint in (3.19) with the first equation in (3.23) yields a second-order ODE for the radial profile. This equation provides an eigenvalue problem that determines the pseudoscalar glueball mass spectrum, $M_n^2$, as well as the corresponding normalizable radial profiles, $h_n, \tilde{h}_n$. Once these profiles are known, the non-derivative $\phi$–$G_n$ couplings arise from the Wess–Zumino term by pulling back $C_7$ onto the D6-branes worldvolume and reducing the result along the $S^2$ and along the radial direction down to four-dmensions. The coordinate $\phi$ is pulled back into a field $\phi_0 + \phi(x)$ that depends *only* on the $\mathbb{R}^4$ coordinates. The form $W_7$ is pulled-back on the ground state of the D6-branes embedding, $\tau = 0, r = r(\lambda)$, since we are only interested in the couplings of the $\eta'$ and not the rest of the mesons. The

non-derivative couplings originate from the first summand in $W_7$, and take the form (setting $\phi_0 = 0$)

$$S_{\rm WZ} \to \frac{N_{\rm f} f_\pi}{2\sqrt{N_{\rm c}}}\, g_n \int_{\mathbb{R}^4} \phi(x)\, G(x) = \sqrt{\frac{N_{\rm f}}{N_{\rm c}}}\, g_n \int_{\mathbb{R}^4} \eta'(x)\, G(x)\,, \tag{3.25}$$

where

$$\frac{f_\pi}{2\sqrt{N_{\rm c}}}\, g_n = \frac{{\rm vol}(S^2)}{(2\pi\ell_s)^6} \int_0^\infty d\lambda\, h_n(U(\lambda))\, r^2 \lambda\, (r - \lambda \dot r)\ , \tag{3.26}$$

and we recall that $\phi$ and $\eta'$ are related as in (1.6). In principle, these couplings can be evaluated numerically, given the embedding $r(\lambda)$ and the eigenmode profiles $h_n(U)$. However, the fact that they are in general non-vanishing is already an important result, for it confirms that the cylinder diagram in figure 5(a) is capable of generating a potential for the $\eta'$ with the right properties.

## 4. A Quantitative Check to Order $1/\sqrt{N_{\rm c}}$

We have argued in the previous sections that certain quantum corrections to the supergravity model with probe D6-branes [1] generate a potential for the $\eta'$ meson of the form

$$V(\eta') = \frac{1}{2}\, \chi_g \left(\theta + \frac{2\sqrt{N_{\rm f}}}{f_\pi}\, \eta'\right)^2 \tag{4.1}$$

to leading order in the $1/N_{\rm c}$ expansion. In this expression, we have considered the $k = 0$ branch of the vacuum energy and we have fixed the additive normalization of the $\eta'$ field so that $V(\eta' = 0)$ equals the pure-glue vacuum energy derived in equation (2.11). With these conventions, taking into account that $f_\pi = O(\sqrt{N_{\rm c}})$, we can expand the square and separate the pure-glue term of $O(1)$, the Witten–Veneziano mass term of $O(N_{\rm f}/N_{\rm c})$, and a cross term of $O(\sqrt{N_{\rm f}/N_{\rm c}})$ which acts as a tadpole upon expanding the potential around the wrong vacuum, $\eta' = 0$. In this section we present a calculation of this linear term by two independent methods, one based on a closed-string calculation plus the anomaly argument, and the other based on a direct open-closed string coupling.

In terms of the $\phi$ field, the 'tadpole' term can be identified as

$${\rm tadpole} = \mathcal{T} = \chi_g\, \theta\, N_{\rm f}\, \phi\ , \tag{4.2}$$

and we may evaluate it in two independent ways. First, we can use the explicit supergravity calculation (2.12) of the pure-glue topological susceptibility and introduce the $\phi$-dependence via the anomaly argument $\theta \to \theta + N_{\rm f}\, \phi$. We find

$$\mathcal{T} = \frac{C\, \theta\, N_{\rm f}}{3 \cdot 2^4 \cdot \pi^5 \cdot \ell_s^6}\, \phi\,, \tag{4.3}$$

where we remind the reader that $C = 3\, U_{\rm KK}^3/\delta\tau$ from (2.10). We emphasize that this calculation only uses the closed-string sector, plus the microscopic anomaly argument.

On the other hand, we may read the linear term directly from the Wess–Zumino action (3.15) for the particular seven-form $C_7$ that is induced by the $\theta$-angle background (2.10). Setting $k = 0$ in this equation, a straight-forward calculation shows that the dual seven-form potential is given (locally) by $C_7 = \phi\, \omega_7$, where

$$\omega_7 = \frac{C\,\theta}{U^4}\, B(U)\,(r\,d\lambda - \lambda\,dr) \wedge d\Omega_2 \wedge dV_4\,, \tag{4.4}$$

where

$$B(U) = \frac{1}{\rho}\,\frac{dU}{d\rho}\,\left(\frac{U}{R}\right)^{9/4}\,\frac{K(\rho)^{3/2}}{f(\rho)^{1/2}}\lambda^2 r\,, \tag{4.5}$$

and we have set to zero the additive normalization of $\phi$, as well as the discrete $2\pi$-shift implementing the large-$N_c$ branches of vacua. Calculating the pull-back of $C_7$ on the D6 world-volume, as in the previous section, we obtain

$$\frac{N_f}{(2\pi\ell_s)^6}\int_{\Sigma_7}\phi\,\omega_7 = \int_{\mathbb{R}^4}\mathcal{T}\,, \tag{4.6}$$

where

$$\mathcal{T} = C\,\theta\,N_f\,\phi\,\frac{\mathrm{vol}(S^2)}{(2\pi\ell_s)^6}\int_0^\infty d\lambda\frac{H(U(\lambda))}{U(\lambda)^4}\;(r - \lambda\dot{r}) \tag{4.7}$$

or, using $\rho^2(\lambda) = r^2(\lambda) + \lambda^2$,

$$\mathcal{T} = \frac{C\,\theta\,N_f}{2^4\cdot\pi^5\cdot\ell_s^6}\,\phi\,\int_0^\infty d\lambda\frac{\lambda^2 r}{\rho^5}(r - \lambda\dot{r})\,. \tag{4.8}$$

Agreement with (4.3) requires the last integral to equal 1/3. Remarkably, this is so, for the integral can be transformed into

$$\int_0^\infty d\lambda\frac{\lambda^2}{\rho^4}\,(\rho - \lambda\dot{\rho}) = \frac{1}{3}\int_0^\infty d\lambda\,\frac{d}{d\lambda}\left(\frac{\lambda^3}{\rho^3}\right) = \frac{1}{3}\,, \tag{4.9}$$

where we have used in the last step the fact that $r(\lambda)$ remains bounded as $\lambda \to \infty$ in the D6-brane embedding.

A more geometrical version of this calculation can be given as follows. We have argued that, locally in the $\phi$-direction, $C_7 = \phi\,\omega_7$. The tadpole comes just from the integration of $\omega_7$ over $\Sigma_7$, the equilibrium worldvolume of the D6-branes at $\phi = 0$. Now, let $\Sigma_8$ be the hypersurface that results from rotating the worldvolume around the angle $\phi$. Since $\partial_\phi\,\omega_7 = 0$ for the $\theta$-induced form (4.4), we can write

$$\int_{\Sigma_7}\omega_7 = \frac{1}{2\pi}\int_{\Sigma_8} d\phi\wedge\omega_7 = \frac{1}{2\pi}\int_{\Sigma_8} F_8\,. \tag{4.10}$$

In addition, $\Sigma_8$ is topologically equivalent to the $S^4$ at fixed $U$ coordinate, times the spacetime $\mathbb{R}^4$ factor. Since $F_8$ is a closed form, we can use Stokes' theorem to write

$$\int_{\Sigma_7}\omega_7 = \frac{1}{2\pi}\int_{\mathbb{R}^4\times S^4} F_8\,. \tag{4.11}$$

The latter integral is trivially evaluated by computing $F_8$ directly in the original coordinate system, in which it takes the simple form

$$F_8 = *F_2 = C\,\theta\, d\Omega_4 \wedge dV_4 \,. \tag{4.12}$$

It follows that the prediction for the tadpole is

$$\mathcal{T} = \phi \cdot \frac{N_{\rm f}}{(2\pi\ell_s)^6} \cdot \frac{C\theta}{2\pi} \cdot {\rm vol}(S^4) = \frac{C\,\theta\,N_{\rm f}}{3 \cdot 2^4 \cdot \pi^5 \cdot \ell_s^6}\,\phi\,, \tag{4.13}$$

again in perfect numerical agreement with (4.3). We regard this check as highly non-trivial, since the kinetic RR term only knows about closed strings (glueballs) and the Wess–Zumino term specifies the direct coupling to the open strings (mesons). The exact agreement for the tadpole is an indication that the basic physical picture is right.

## 5. Concluding Remarks

In [1] a string dual of large-$N_{\rm c}$ QCD with $N_{\rm f}$ flavours, based on $N_{\rm f}$ D6-brane probes in a fixed supergravity background, was studied in detail. It was found that the string description captures some of the low-energy physics expected on field theory grounds. In particular, for $N_{\rm f} = 1$, it exhibits spontaneous chiral symmetry breaking of an $U(1)_{\rm A}$ symmetry, and the mesonic spectrum contains a pseudoscalar that is exactly massless at infinite $N_{\rm c}$. This is the analog of the $\eta'$ meson of large-$N_{\rm c}$ QCD, and is dual to the zero-mode associated to the motion of the D6-brane.

As discussed in the Introduction, the importance of the Witten-Veneziano formula (1.7) is in producing a qualitative understanding of the mass splitting of the $\eta'$ meson from the other light mesons, namely the pseudo-Goldstone modes associated with the spontaneous breaking of the chiral flavour symmetry. With multiple flavours, that is, multiple D6-branes, our AdS-like model produces $N_{\rm f}^2$ massless pseudoscalars [1]. However, only the diagonal mode corresponding to the collective center-of-mass motion of all the D6-branes is obviously a Goldstone mode. In Appendix B, we argue that, when the analysis is taken beyond tree-level, the other $N_{\rm f}^2 - 1$ modes acquire masses of order $(\lambda N_{\rm f}/N_{\rm c})^{1/2} M_{\rm KK}$, where $\lambda = g_{\rm YM}^2 N_{\rm c}$ is the 't Hooft coupling. As further shown in Appendix B, this precisely matches the mass of the $\eta'$ at the level of their parametric dependences. We interpret the masses of the off-diagonal modes as arising from closed string interactions between the individual D6-branes and so, to leading order, they are generated by the same string diagrams as illustrated in figure 5(b) – recall only the diagonal mode couples in figure 5(a). This result points to a qualitative distinction between the physics of our model and QCD: even in the limit of vanishing quark masses, our entire multiplet of $N_{\rm f}^2$ light mesons acquires a mass squared of the same order as the $\eta'$, while in QCD only the $\eta'$ becomes massive. The additional mass terms in the present case are natural as the off-diagonal scalars are not Goldstone modes — in fact, it is the masslessness of these modes in the large-$N_{\rm c}$ limit that was surprising [1]. However, it would still be interesting to refine the estimates made in Appendix B, to see if there is

any dramatic difference in the numerical values of the off-diagonal and the $\eta'$ masses for our model.

On general grounds, the identification by Witten [9] of an $O(1)$ contribution to the pure-glue topological susceptibility, plus a microscopic anomaly argument, implies the generation of a potential for the $\eta'$ with a mass term of $O(N_{\rm f}/N_{\rm c})$, along the lines of the Witten–Veneziano argument. We have argued that the cylinder diagram of figure 5(a) is the relevant stringy correction responsible for the generation of the $\eta'$ mass, in a string analog of the Isgur-de Rújula-Georgi-Glashow mechanism [8]. We have shown that, in the supergravity approximation, this diagram induces non-derivative mixings between the glueballs and the $\eta'$ that shift the zero-momentum pole of the $\eta'$. However, this shift by itself would make the $\eta'$ tachyonic, and hence we argued that there is an important contact term coming from the stringy completion of the glueball-exchange diagrams in figure 5(a). This discussion is natural when one thinks of the worldsheet cylinder as being long in comparison to its circumference. Coming from the opposite end of the worldsheet moduli space (*i.e.,* a short cylinder with a large circumference), this diagram has a natural interpretation, depicted in figure 7(a), in terms of an open string, and hence meson, loop, where the internal meson propagators are now both twisted. From this point of view, which naturally figures as the UV completion of the glueball sum, these loop contributions do not have a definite sign and so certainly allow for the necessary shift with a positive sign. Of course, string duality tells us that that the sums over all tree-level glueball exchanges and over one-loop meson graphs are the same and should not be computed separately.

Similarly, the open string or meson loop of figure 7(b) also has an interpretation as a closed string coupling with two $\eta'$ mesons and being absorbed by the D6-brane, as displayed in figure 6(b). It might be emphasized here that both diagrams in figure 6 play an important role in the $\eta'$ physics. This is particularly evident from the discussion at the beginning of section 3.3, where we argue that the Neveu-Schwarz glueballs will *not* generate a potential for the $\eta'$. To see how the cancellation presented there occurs order by order in $\phi$, one would make a Taylor expansion of the exponentials in equation (3.11). At order $\phi^2$, this reveals that the vanishing mass arises precisely as a cancellation between glueball exchange as in figure 6a and tadpole contributions of figure 6(b).

More generally, the anomaly argument implies that the dependence of the theory on the $\theta$-angle occurs only through the combination $\theta + 2\sqrt{N_{\rm f}}\,\eta'/f_\pi$, so that the microscopic $\theta$-dependence can be eliminated by a $U(1)_{\rm A}$ transformation. We have verified this statement in the ultraviolet regime by analizing the RR flux sourced by the D6-branes.

We have strenghened the physical picture by performing a quantitiative check of the $\eta'$-potential at order $1/\sqrt{N_{\rm c}}$. We have computed this potential in two independent ways. One method employs only the closed string sector, that is, the pure-glue sector, together with the anomaly argument. The second method involves the open string sector, that is, the mesonic sector. We regard the perfect agreement between the two results as a non-trivial check that the right physics is captured.

The 'master substitution' $\theta \to \theta + 2\sqrt{N_{\rm f}}\,\eta'/f_\pi$ applied to the pure-glue effective La-

grangian generates all soft-$\eta'$ amplitudes. In the supergravity formalism, this implies precise correlations between the effective couplings of the closed string sector, and those of the closed string sector to the open string sector. The simplest of these correlations was checked in section 4, but it would be interesting to investigate the more complicated ones, even at a qualitative level.

Since the pure-glue $\theta$-dependence comes from the energy of RR fluxes, the stringy mechanism is akin to a Green–Schwarz modification of the RR field strengths.[10] It would be interesting to sharpen the anomaly argument in the supergravity regime by identifying a ten dimensional anomaly polynomial that yields the substitution $\theta \to \theta + 2\sqrt{N_{\rm f}}\,\eta'/f_\pi$ as a standard Green–Schwarz modification of the $F_2$ field strength, after appropriate reduction on the D6-branes worldvolume. In this case, the basic stringy calculation of the ultraviolet contact terms could be performed locally in the flat limit of the ten-dimensionsional string theory.

## Acknowledgments

We would like to thank Margarita García Pérez, César Gómez and Sean Hartnoll for helpful discussions, and especially Adi Armoni for many discussions on the subject of this paper and [16]. The work of J.L.F.B. was partially supported by MCyT and FEDER under grant BFM2002-03881 and the European RTN network HPRN-CT-2002-00325. The work of C.H. was partially supported by European Commission (HPRN-CT-200-00148) and CICYT (Spain) and by the MECD (Spain) through a FPU grant. Research at the Perimeter Institute is supported in part by funds from NSERC of Canada. RCM is further supported by an NSERC Discovery grant.

## A. Large-$N_{\rm c}$ Scalings from Supergravity and DBI

In the presence of $N_{\rm f}$ D6-branes, the low-energy dynamics on the gravity side is described by the action $S = S_{\rm Sugra} + S_{\rm D6}$. Let $\Phi$ and $\phi$ denote collectively fluctuations of the supergravity and the D6-branes' worldvolume fields, respectively. Schematically, the action for these fields takes then the form

$$S = N_{\rm c}^2 \int_{M_{10}} (\partial\Phi)^2 + \sum_{\ell\geq 2} \Phi^\ell + N_{\rm f} N_{\rm c} \int_{\Sigma_7} (\partial\phi)^2 + \sum_{m\geq 0, n\geq 1} \Phi^m \, \phi^n \, . \tag{A.1}$$

The coefficients in front of each integral arise from the scalings with $g_s$ of Newton's constant, $G_N \sim g_s^2$, and the fact that $g_s \sim 1/N_{\rm c}$ in the 't Hooft limit. Further the D6-branes' tension scales as $T \sim N_{\rm f}/g_s$ — note that we are assuming that $\phi$ is a $U(1)$ field in the $U(N_{\rm f})$ gauge group on the D6-branes, as is the $\eta'$.

In terms of canonically normalized fields, defined through $\Phi \to N_{\rm c}\Phi$ and $\phi \to \sqrt{N_{\rm f} N_{\rm c}}\phi$, we find

$$S = \int_{M_{10}} (\partial\Phi)^2 + \sum_{\ell\geq 2} N_{\rm c}^{2-\ell}\Phi^\ell + \int_{\Sigma_7} (\partial\phi)^2 + \sum_{m\geq 0, n\geq 1} N_{\rm f}^{1-\frac{n}{2}} N_{\rm c}^{1-m-\frac{n}{2}} \Phi^m \, \phi^n \, . \tag{A.2}$$

[10] See [16] for a study of this interpretation in a slightly different model.

The strength of the different couplings can now be directly read off. For example, a glueball tadpole (*i.e.*, a closed string one-point function) is of order $N_{\rm f}$ while the $\eta'$-glueball coupling is of order $\sqrt{N_{\rm f}/N_{\rm c}}$. These results are used in section 3.2.

## B. Pseudoscalar Masses Revisited

In this paper, we have argued that the $\eta'$ meson, in the model of [1], acquires a mass consistent with the Witten-Veneziano formula:

$$m_{\eta'}^2 = \frac{4N_{\rm f}}{f_\pi^2}\,\chi_g\,. \tag{B.1}$$

We now wish to evaluate this mass in terms of the microscopic parameters of the field theory. In the following, we will only determine the parametric dependence but drop numerical factors. First, equation (2.12) gives the topological susceptibility as

$$\chi_g \sim \lambda^3 M_{\rm KK}^4 \tag{B.2}$$

where $\lambda = g_{\rm YM}^2 N_{\rm c}$ is the 't Hooft coupling. From (4.18) of [1], one deduces that, up to numerical factors, the pion decay constant is given by

$$f_\pi^2 \sim T_{\rm D6} U_{\rm KK}^3/M_{\rm KK}^2 \sim N_{\rm c}\lambda^2 M_{\rm KK}^2\,. \tag{B.3}$$

The second expression above was determined using (2.8) and $T_{\rm D6} \sim 1/g_s \ell_s^7$. Combining these results, we find

$$m_{\eta'}^2 \sim \frac{N_{\rm f}}{N_{\rm c}}\,\lambda\, M_{\rm KK}^2\,. \tag{B.4}$$

With a collection of $N_{\rm f}$ D6-branes in the holographic model of [1], the dual theory contains $N_{\rm f}$ quark flavors. If all of the quark masses vanish, it was shown to leading order in the large-$N_{\rm c}$ expansion that the spectrum contains $N_{\rm f}^2$ massless pseudoscalar mesons. Of these, only the $\eta'$ meson, which corresponds to the collective motion of all of the D6-branes together, appears to be a true Goldstone mode. Again the latter only applies in the large-$N_{\rm c}$ limit, as we have argued here that a mass appears through $1/N_{\rm c}$ effects. Similarly it was argued in [1] that the remaining $N_{\rm f}^2-1$ pseudoscalars will acquire masses at this order. These mesons can be interpreted as modes which separate the individual D6-branes and so the masses can be understood as arising from closed string interactions between the separated branes. We will now give a concrete (albeit crude) estimate of the masses which are produced in this way. In the following, we will be working with the background D4-brane metric in the form given in eq. (2.16).

Imagine we have embedded $N_{\rm f}$ D6-branes with $m_{\rm q}=0$ but they have been divided into two groups of $N_{\rm f}/2$ branes separated by a small angle $\delta\phi$ in the 89-plane, as depicted in figure 8. We will assume this depicts a canonical mode and so its mass is typical of that for all of the off-diagonal modes. We will also assume that the interaction can be approximated by integrating the 'Newtonian' potential between volume elements on the separated (sets of)

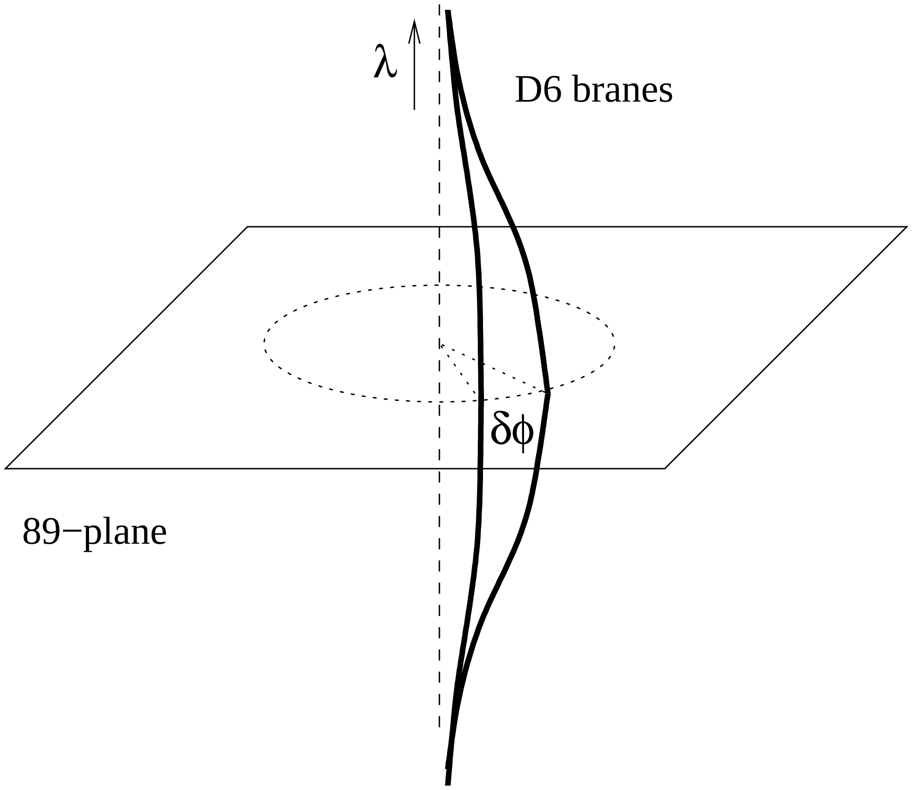


**Figure 8:** D6-brane embedding for $U_{\rm KK} \neq 0$ and $m_{\rm q} = 0$, but separating two groups of $N_{\rm f}/2$ D6-branes by a small angle $\delta\phi$ in the 89-plane.

branes. Now as all of the branes have the 123-space in common, it is simplest to consider integrating over slices which extend through this subspace. These three-dimensional volume elements are then only infinitesimal in their extent in the $z^i$ subspace where the embedding is nontrivial, and will interact with a $1/\rho^4$ potential. We express the interaction energy as the energy density per unit volume (in the 123-space), which will roughly take the form:

$$\delta\mathcal{E} \simeq N_{\rm f}^2\, G_{10}\, h(\Omega_1,\Omega_2)\frac{(T_{\rm D6}\,\delta^3A)_1\,(T_{\rm D6}\,\delta^3A)_2}{|\vec{z}_1-\vec{z}_2|^4}\ , \tag{B.5}$$

where we have introduced $h(\Omega_1,\Omega_2)$ to indicate that the strength of the interaction depends on the details of the relative orientation of the two area elements.[11]

Now if we examine eq. (2.17) of [1] which determines the embedding profile of the D6-branes in the background geometry, we see that $U_{\rm KK}$ is the only scale which enters this equation and so this is the scale of the interesting deformation in figure 8. Hence, we expect

[11] Implicitly we will assert below that this interaction strength is positive. This intuition comes from considering D-branes in flat space where static parallel branes are supersymmetric. Hence a relative rotation between elements of the brane can only increase the energy.

that the appropriate integrations yield

$$\left\langle h(\Omega_1,\Omega_2)\frac{(\delta^3 A)_1\,(\delta^3 A)_2}{|\vec{z}_1-\vec{z}_2|^4}\right\rangle \sim U_{\rm KK}^2\,\delta\phi^2\ , \tag{B.6}$$

up to an overall purely numerical factor. That is, $U_{\rm KK}$ and $\delta\phi$ are the only relevant scale and angle[12], respectively, in the geometry of the D6-brane configuration. However, we must also account for the fact that all proper distances in the $z^i$ subspace are contracted by a factor of $K^{1/2}$, as seen in eq. (2.16). Hence the energy density acquires an additional factor of

$$K=\frac{R^{3/2}U^{1/2}}{\rho^2}\sim\left(\frac{R}{U_{\rm KK}}\right)^{3/2}\ , \tag{B.7}$$

where in the last expression we have again used the approximation that $U_{\rm KK}$ is the only scale relevant for the embedding. Next, we note that the final expression may be simplified using

$$G_{10}\,T_{\rm D6}^2\simeq g_s^2\ell_s^8\left(1/g_s\ell_s^7\right)^2=1/\ell_s^6\ . \tag{B.8}$$

Hence our approximation for the total interaction energy density becomes

$$\begin{aligned}\mathcal{E}\,&\sim\,N_{\rm f}^2\frac{R^{3/2}U_{\rm KK}^{1/2}}{\ell_s^6}\,\delta\phi^2\\ &\sim\,N_{\rm f}^2\frac{\lambda}{\ell_s^4}\,\delta\phi^2\end{aligned} \tag{B.9}$$

where we simplified the second line with (2.8).

Now one might find the appearance of $\ell_s$ in this expression disturbing, however, one must realize that this is not a field theory quantity rather it is a proper energy density, $\mathcal{E}_{proper}$. To convert our expression to the energy density in the dual field theory, we must include additional metric factors for the $x^\mu$ directions relating proper bulk space distances to simple coordinate distances, which are relevant for the field theory, *i.e.*, from eq. (2.16), $\Delta x_{proper}\sim(U_{\rm KK}/R)^{3/4}\,\Delta x_{coord}$ again making the approximation that the only scale relevant for the embedding is $U_{\rm KK}$. Hence we have

$$\mathcal{E}_{field}\sim\left(\frac{U_{\rm KK}}{R}\right)^3\mathcal{E}_{proper}\sim N_{\rm f}^2\lambda^3M_{\rm KK}^4\,\delta\phi^2 \tag{B.10}$$

where the latter expression is simplified using (2.8). Finally to identify the mass, we must normalize the coefficient of $\delta\phi^2$ above by that in front of the corresponding kinetic term. Following the discussion of [1], it is straightforward to see that the kinetic term for this off-diagonal mode has an overall factor of $N_{\rm f}f_\pi^2$. Hence, using eq. (B.3), our final expression for the mass is

$$m_{pseudo}^2\sim\frac{N_{\rm f}}{N_{\rm c}}\,\lambda\,M_{\rm KK}^2\,. \tag{B.11}$$

Hence the parametric dependence of these off-diagonal pseudoscalar masses precisely matches that of the $\eta'$ in eq. (B.4). It may be interesting to perform a more detailed analysis to determine the numerical coefficients in these two mass formulae.

[12]Symmetry would rule out the appearance of a single power of $\delta\phi$ and hence the leading contribution must be $\delta\phi^2$.

## References


[1] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, *Towards a holographic dual of large-$N_c$ QCD*, *J. High Energy Phys.* **05** (2004) 041, `hep-th/0311270`.

[2] M. Gell-Mann, R. J. Oakes and B. Renner, *Behavior of current divergences under $SU(3) \times SU(3)$*, *Phys. Rev.* **175** (1968) 2195.

[3] S. Weinberg, *The $U(1)$ Problem*, *Phys. Rev.* **D 11** (1975) 3583.

[4] G. 't Hooft, *Symmetry Breaking through Bell-Jackiw Anomalies*, *Phys. Rev. Lett.* **37** (1976) 8; *Computation of the Quantum Effects due to a Four-dimensional pseudo-particle*, *Phys. Rev.* **D 14** (1976) 3432; *How Instantons Solve the $U(1)$ Problem*, *Phys. Rev.* **142** (1986) 357.

[5] E. Witten, *Current Algenra Theorems for the $U(1)$ Goldstone Boson*, *Nucl. Phys.* **B 156** (1979) 269.

[6] G. Veneziano, *$U(1)$ without Instantons*, *Nucl. Phys.* **B 159** (1979) 213.

[7] E. Witten, *Large-N Chiral Dynamics*, *Ann. Phys. (NY)* **128** (1980) 363; P. Di Vecchia and G. Veneziano, *Chiral Dynamics in the large N limit*, *Nucl. Phys.* **B 171** (1980) 253; C. Rosenzweig, J. Schechter and G. Trahern, *Is the Effective Lagrangian for QCD a Sigma Model?*, *Phys. Rev.* **D 21** (1980) 3388; K. Kawarabayashi and N. Ohta, *The $\eta$ Problem in the Large-N Limit: Effective Lagrangian Approach*, *Nucl. Phys.* **B 175** (1980) 477; H. Hata, T. Kugo and N. Ohta, *Skew-Symmetric Tensor Gauge Field Theory Dynamically Realized in the QCD $U(1)$ Channel*, *Nucl. Phys.* **B 178** (1981) 527; P. Nath and R. Arnowitt, *The $U(1)$ Problem: Current Algebra and the Theta Vacuum*, *Phys. Rev.* **D 23** (1981) 473.

[8] A. De Rújula, H. Georgi and S. L. Glashow, *Hadron Masses in a Gauge Theory*, *Phys. Rev.* **D 12** (1975) 147; N. Isgur, *Mass Formula for Nonets*, *Phys. Rev.* **D 13** (1976) 122.

[9] E. Witten, *Theta-dependence in the Large-N Limit of Four-dimensional Gauge Theories*, *Phys. Rev. Lett.* **81** (1998) 2862, `hep-th/9807109`.

[10] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505, `hep-th/9803131`.

[11] R. C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022, `hep-th/9910053`.

[12] O. Aharony, A. Fayyazuddin and J. M. Maldacena, *The large-N limit of $\mathcal{N} = 2, 1$ field theories from threebranes in F-theory*, *J. High Energy Phys.* **07** (1998) 013, `hep-th/9806159`; A. Karch and L. Randall, *Open and closed string interpretation of SUSY CFT's on branes with boundaries*, *J. High Energy Phys.* **06** (2001) 063, `hep-th/0105132`; M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, and R. Marotta, *N=2 Gauge theories on systems of fractional D3/D7 branes*, *Nucl. Phys.* **B 621** (2002) 157, `hep-th/0107057`; M. Bertolini, P. Di Vecchia, G. Ferretti, and R. Marotta, *Fractional Branes and N=1 Gauge Theories*, *Nucl. Phys.* **B 630** (2002) 222, `hep-th/0112187`; A. Karch and E. Katz, *Adding flavor to AdS/CFT*, *J. High Energy Phys.* **06** (2002) 043, `hep-th/0205236`;

[13] A. Karch, E. Katz and N. Weiner, *Hadron masses and screening from AdS Wilson loops*, *Phys. Rev. Lett.* **90** (2003) 091601, `hep-th/0211107`; M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, *Meson spectroscopy in AdS/CFT with flavour*, *J. High Energy Phys.* **07** (2003) 049, `hep-th/0304032`; T. Sakai and J. Sonnenschein, *Probing flavored mesons of confining gauge*

*theories by supergravity*, *J. High Energy Phys.* **09** (2003) 047, hep-th/0305049; J. Babington, J. Erdmenger, N. Evans, Z. Guralnik and I. Kirsch, *Chiral symmetry breaking and pions in non-supersymmetric gauge/gravity duals*, *Phys. Rev.* **D 69** (2004) 066007, hep-th/0306018; X. J. Wang and S. Hu, *Intersecting branes and adding flavors to the Maldacena-Nunez background*, *J. High Energy Phys.* **09** (2003) 017, hep-th/0307218; P. Ouyang, *Holomorphic D7-branes and flavored N = 1 gauge theories*, hep-th/0311084; C. Nunez, A. Paredes and A. V. Ramallo, *Flavoring the gravity dual of $\mathcal{N} = 1$ Yang-Mills with probes*, *J. High Energy Phys.* **12** (2003) 024, hep-th/0311201; S. Hong, S. Yoon and M. J. Strassler, *Quarkonium from the Fifth Dimension*, *J. High Energy Phys.* **04** (2004) 046, hep-th/0312071; E. Schreiber, *Excited Mesons and Quantization of String Endpoints*, hep-th/0403226; N. Evans and J. P. Shock, *Chiral Dynamics From AdS Space*, hep-th/0403279.

[14] A. Hashimoto and Y. Oz, *Aspects of QCD Dynamics from String Theory*, *Nucl. Phys.* **B 548** (1999) 167, hep-th/9809106.

[15] J.L.F. Barbón and A. Pasquinucci, *Aspects of Instanton Dynamics from AdS/CFT duality*, *Phys. Lett.* **B 458** (1999) 288, hep-th/9904190.

[16] A. Armoni, *Witten–Veneziano from Green–Schwarz*, *J. High Energy Phys.* **06** (2004) 024, hep-th/0404248.